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ETHNOMATHEMATICS AND EVERYDAY COGNITION

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The purpose of this chapter is to look at current research on culture and mathematics learning and to use this research in the analysis of the acquisition of mathematical concepts and skills. The chapter is divided into five parts. The first section briefly contrasts two views of cultural influences on mathematical activities. The next three sections discuss traditional issues in mathematics education: (1) counting and measuring, (2) solving arithmetic calculations, and (3) modeling and knowledge of inverse operations. These topics do not cover all types of mathematical activities. They were chosen because of their central role in elementary mathematics education, and because current work on these topics generates an interesting picture of similarities and differences in mathematical knowledge used in distinct cultural situations. The final section turns to theoretical concepts and educational implications of research on culture and mathematics.

TWO VIEWS OF CULTURE AND MATHEMATICS

For years mathematics educators and researchers in mathematics education have focused on the classroom as the primary setting in which mathematics learning takes place. More recently, survey studies (for example, Cockcroft, 1986) as well as analyses of children's knowledge of mathematics (for example, Carraher, Carraher, & Schliemann, 1985, 1987; Ginsburg, 1977; Ginsburg, Posner, & Russel, 1981; Hughes, 1986; Resnick, 1984) have shown that much mathematical knowledge is acquired outside school. The realization that mathematical knowledge can be acquired outside school brings new variables into the analysis of mathematics learning and teaching. Is the mathematics learned outside school the same as that taught in school? How can teachers identify and capitalize on mathematics learned outside school?

D'Ambrosio (1984, 1985) has used the expression "ethnomathematics" to refer to forms of mathematics that vary as a consequence of being embedded in cultural activities whose purpose is other than "doing mathematics." Everyday activities such as building houses, exchanging money, weighing products, and calculating proportions for a recipe involve numbers, calculations, and precise geometrical patterns. These applications of mathematics often look different from those used in school. In the kitchen we often measure volume with spoons and cups, whereas in school activities students typically measure volume in liters or cubic meters. Everyday mathematics also varies significantly across countries, because of differences in the numeration systems used, for example, or the devices used for calculating. These differences may be perceived as deep- or surface-structure differences, depending on what views one holds of mathematical knowledge.

Two distinct approaches to the study of cultural influences on mathematical knowledge can be identified in the current

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literature. One view, espoused by Stigler and Baranes (1988), holds that

> Mathematics is not a universal, formal domain of knowledge . . . but rather an assemblage of culturally constructed symbolic representations and procedures for manipulating these representations. . . . As children develop, they incorporate representations and procedures into their cognitive systems, a process that occurs in the context of socially constructed activities. *Mathematical skills that the child learns in school are not logically constructed on the basis of abstract cognitive structures* [italics added], but rather are forged out of a combination of previously acquired (or inherited) knowledge and skills, and new cultural input. (p. 258)

In this approach, the definition of mathematical knowledge is implicit and appears to be based on the content of knowledge. Activities that involve number, geometrical patterns, calculation, and so forth are treated as applications of mathematical knowledge. This view stresses differences rather than similarities across cultures. Within this approach, for example, variations across cultures of reciting strings of numbers are treated as a reflection of the differences in language and numeration systems. A second perspective, illustrated in the works by Gal'perin and Georgiev (1969) and D'Ambrosio (1986), suggests that the analysis of cultural influences on mathematical knowledge can demonstrate both differences and invariance in mathematical knowledge across cultures. In this view, "mathematizing" reality is representing reality in such a way that (a) more knowledge about the represented reality can be generated through inferences using mental representations, and (b) there is no need to manipulate reality further in order to verify this new knowledge. Invariant logical structures are embedded in mathematical knowledge, regardless of whether mathematical knowledge is developed in or out of school. It is the ability to make inferences using these structures—not the content of knowledge—that distinguishes mathematical knowledge. Merely reciting count words, for example, would not be considered mathematical knowledge in this perspective. Despite the fact that numbers are the content of knowledge in this case, no inferences result from mere recitation, thus no mathematical knowledge is involved. On the other hand, if count words are used to represent sets and make inferences about relationships between sets, the count words bear on mathematical knowledge. Different cultures have found distinct solutions to the organization of their count words. These differences have important effects on how quickly children learn the count words, as Miller and Stigler (1987) have shown in their comparisons of Chinese and American children. However, invariant principles underlie the activity of counting objects in both cultures (counting principles will be discussed later in this paper), and the inferences that are made by Chinese and American children using number words as representations of quantity are the same despite the differences in the words used.

In my view, understanding cultural influences on mathematics learning must involve both the analysis of the differences among particular solutions to mathematizing reality and the recognition of the logical invariants underlying these differences. Cultural differences exist not only across cultures. Within a culture, practices of mathematics vary depending on their purpose. Research on mathematics education can benefit from the analysis of various solutions to the same problem that co-exist in a single culture. In cultures where there are both written and oral representations of number, it is likely that two practices of arithmetic co-exist (Reed & Lave, 1981). Research may show that two varieties of mathematical practice within the same culture differ in some ways but are similar in others. The similarities are likely to relate to the logical invariants of mathematical knowledge. Any variations are likely to relate to the representations and the uses of the invariants in the mathematical applications. These differences are peripheral to the conceptual structure but central to the skills displayed by the subjects.

THE ETHNOMATHEMATICS OF COUNTING AND MEASURING: CONTRIBUTIONS OF CULTURE AND LOGIC

In this section, the logical invariants involved in counting will be examined, as well as cultural variations for overcoming the difficulty of applying counting principles.

Counting Principles and Cultural Contributions to the Development of Counting Systems

Counting and measuring are ways of representing selected aspects of objects and situations. In order to measure, one has to choose what dimension will be quantified—a class of objects when counting, for example, or a quality like length or weight when measuring. Like any form of representation, the initial choice of what to represent and what to ignore involves an abstraction: Everything will be ignored except the aspect that is being quantified. The activities of counting or measuring are usually carried out for some larger purpose. You count the money in your pocket to determine whether you have enough for a particular purchase. You measure a table to determine the size of the tablecloth that will cover it properly. Carrying out such activities is what makes counting and measuring meaningful.

The activities in which we use counting and measuring vary, yet an underlying logic seems present. In a thorough analysis of how children develop counting skills, Gelman and Gallistel (1978) specified four basic logical principles that must be satisfied if an activity is to be classified as counting. These principles may be summarized as (1) establishing a one-to-one correspondence between the things to be counted and the counting labels, (2) maintaining the counting labels in a fixed order, (3) recognizing the irrelevance of the order in which the objects are counted, and (4) applying the cardinality principle, that is, using the last label to represent the number of objects in the set. These four principles have such a strong logical appeal that Gelman and her collaborators are willing to treat them as culturally independent and possibly innate.

A system based only on these logical principles has limited application. In the absence of a culturally organized numeration system, one would have difficulty obeying these principles. How many labels can one remember in a fixed order (principle 2) if the labels are unrelated words and are not part of a system that makes their production easy? The human capacity to memorize ordered lists is limited in the absence of some structural support. Without a numeration system, counting would be restricted to low numbers; with a numeration system, it can go on indefinitely.

Although these counting principles may be culturally independent, specific numeration systems are culturally dependent. The important work of Lancy (1983), Saxe (1981), and others clarifies how different cultures have addressed the problem of memory load in counting. The Kewa and the Oksapmin of Papua New Guinea have developed numeration systems that help them maintain fixed order by using the names of body parts as labels in counting. For example, *thumb* indicates *one, index two, middle-finger three,* and so on. The use of body parts in counting is a cultural and conventional solution to the problem of memory load: The parts to be named and the order in which they are used must be agreed on. Some of the body parts chosen do not have clearly identifiable labels in many Western cultures, such as the three locations on the forearm and six locations between the shoulder and the neck. This use of body parts labeled systematically allows the Kewa to count to 68.

This is by no means the only or the best solution to this problem. In English another solution for the problem of memory load uses a base system in generating number labels. The count words used in English are maintained in the proper order by their generation in systematic combinations. Number words are unrelated to each other up to *twelve*, but from *thirteen* on there are cues in the labels that help generate the succeeding labels in a fixed order. These cues are even clearer after *twenty-one* when labels are used recursively to produce derivative count words. The generation of count words in this fashion is related to the introduction of a base in the counting system.

A base in a numeration system is a grouping scheme used to reorganize counting. To define a base in a numeration system is to choose a conventional unit (several conventional units for a mixed-base system) to be used in counting. According to Luria (1969), a base-numeration system involves counting natural objects, organizing them in conventional groups that become new counting units, and grasping the semantically complex structure underlying the numeration system. The number 343 expresses that there are three groups of one-hundred, four groups of ten, and three objects. In order to capture this meaning and generate number labels indefinitely, subjects must not think of the natural object as the only thing being counted; they must also understand the structure of meaning in the numeration system. Luria attributed a great importance to this distinction, arguing that the counting of natural objects without an understanding of the base system is carried out by the brain's right hemisphere, whereas understanding of the underlying base system is controlled by the left hemisphere.

In summary, counting systems rest both on a logic of invariant principles and on culturally specific devices for the implementation of these principles. Not all cultures find the same solution for the challenge of keeping number labels in a fixed order. The different solutions vary in their ability to deal with large numbers. The systematic enumeration of body parts is one type of solution for keeping count words in a fixed order, but this device has limited range. The use of a base in a numeration system is a cultural device that can solve memory overload problems in counting. A base system allows for counting indefinitely, an impossibility with the nonbase, body-parts systems. From the psychological viewpoint, the introduction of a counting base involves the concept of counting-units. In a base system counting-units are not only the natural objects but also the conventional groups of objects indicated by the base in the system. Despite the culturally dependent nature of the resource we call a base, the use of counting-units is not simply conventional and devoid of logic. A base numeration system is supported by the concept of unit, which is common to both counting and measurement.

Cultural Variation and the Underlying Logic of the Concept of Units

There are several contexts in which the concept of units is used in everyday activities in Western cultures. In the numeration system we have counting-units of ones, tens, hundreds, and so on. For measurement we use the metric system. In monetary exchange we use coins and notes of various values. In each of these contexts, the concept of unit is present. But can it be effectively learned in everyday practice without systematic instruction?

Two types of study of everyday cognition and the use of units in counting and measuring will be discussed. The first describes the emergence of counting groups that model what is counted. The second investigates inference-making about units of different sizes.

Reinventing Counting-units for Particular Purposes. Two studies carried out with subjects from different parts of the world indicate that people can reinvent the concept of units in the context of everyday activities. These studies were conducted by Saxe (1982; 1985) among the Oksapmin and by Scribner and her colleagues (Fahrmeier, 1984; Scribner, 1984) among workers in a dairy factory in the United States.

Saxe (1982) described how a new social activity, the emergence of a money economy, influenced the Oksapmin's use of their existing numeration system. The Oksapmin had a no-money economy in the past, therefore they had little need for computation before their contact with the West. The emergence of a money economy began through their contacts with missions and farms, and marked changes were observed in the way that adults who became involved with commerce used their indigenous system. Saxe described the following changes:

> Since only limited quantities can be expressed with the Oksapmin system, communicating about even small denominations of currency presents serious problems. The adaptation that has emerged is one that incorporates the base structure of the early Australian currency system into the indigenous system. With the adapted system, rather than using all 27 body parts in an enumeration, an individual counts shillings up to the inner elbow on the other side of the body (20) and calls it 1 round, or 1 pound (reflecting the organization of the first Australian currency system). If the individual needs to continue the count, he or

she begins again at the thumb of the first hand (rather than progressing on the forearm [21]).... The adapted system, then, has a base structure that reflects the base structure of the early Australian currency system but nevertheless is an outgrowth of the standard indigenous system. (p. 585)

This reinvention of a grouping unit that is introduced into the indigenous system and that parallels the grouping unit of the exogenous monetary system constitutes a clear example of modeling.

The work by Scribner and her colleagues investigated the use of various methods of grouping for counting that are shaped by everyday activity. Fahrmeier (1984) described inventory-taking in a milk factory in the United States, which required assessment of quantities of some 100 products stored in a walk-in refrigerator.

Counts for each product need to be accurate within a 1% or 2% margin of error. Larger errors result either in shortages or over-production, both costly events.... Products were stored so close to each other that inventory men had limited walk room for maneuvering around arrays. They had to seek out and reach vantage points from which a count could be made, often doing this by climbing on cases to "see over" the front row of an array. By this move, they could almost always see the top cases of the stacks but not the cases underneath. For much of the time, then, they were taking counts of arrays containing invisible cases. (p. 7)

As Fahrmeier points out, these circumstances made counting different from putting items and number labels into one-to-one correspondence. Although the counting principles have to be honored for a correct count to be reached, they cannot be carried out literally. Fahrmeier describes five strategies that emerged in taking inventory under these circumstances, all of which involved the use of new units based on spatial groupings—stacks with known height in terms of number of cases. By representing the cases in stacks of known heights and using the stacks as the new unit in counting, the problem became soluble within the limits of time and accuracy required by the job. A traditional system of counting was adapted to better handle the requirements of the activity by incorporating the properties of stacks as a base for the counting procedure. Scribner (1984) describes similar uses of a new unit of grouping for product assemblers in the same factory for whom the unit was a case with 16 quarts, a number that requires awkward maneuvers (like carrying) when it is repeatedly added in the decimal numeration system.

Inference-making about Units. It is possible to go further in the study of conventional units than looking simply at counting. One can design tasks that test whether people make consistent mathematical inferences when thinking about units in these everyday practices. Four examples of inference-making about units are presented below, first in a formal description and then in the context of research.

The first example is a case of simple transitive inference: (1) If $A = B$ and $B < C$, then $A < C$.

A second example of inference-making about units is an extension of a relationship between two units and any equal number of those two units: (2) If A and B represent different units of measurement of some variable and $A > B$, then for every positive number x, $xA > xB$.

These two examples refer to the logic of units and do not involve a precise quantification of the relationship between the units. The next examples deal with the quantification of relationships between units. The first may seem trivial, but it is not simple for children (see Carraher & Schliemann, 1990):

> *(3) If* A *and* B *represent different units of measurement of some variable and* A = xB, *then* A > (x − *1*)B *and this is true despite the greater numerosity of* (x − *1*)B.

The last example of inference-making using the quantification of the relationships between units involves the recognition of the additive composition of measurement:

> *(4) If* A *and* B *represent different units of measurement of some variable and* A = xB, *then any value of the variable measured in* As *can be expressed in terms of* B *values, and any* B *values greater than* A *can be expressed in terms of* A *values plus the remaining* Bs.

Saxe and Moylan (1982) analyzed the ability of Oksapmin children and adults to make type 1 inferences and the influence of schooling on this ability when the children were using their indigenous measurement system for determining length. Their system of using body parts to represent measurements depends on the size of the body parts, which varies across subjects. Oksapmin adults and children five and older wear string bags that are measured by putting an outstretched hand into the bag and describing its expanded size as *knuckles, wrist, forearm, inner elbow, biceps,* or *shoulder.*

This measurement system creates problems in inference-making. The same bag may be measured differently by a child and an adult. Saxe and Moylan set out to investigate whether Oksapmin children and unschooled adults understood the variability of their units and whether schooling contributed to this understanding.

They gave their subjects two groups of tasks. In the first group, the unit of measurement (body part) was held constant. Subjects had to judge whether bags A and C, which they never compared directly, were the same size by measuring them against their own body part B. In these tasks, the problem structure was one of transitive inference: $A = B$; $C > B$; thus $C > A$. In the second group of tasks, the unit of measurement varied. Subjects were told that a bag had been measured on a child, and they were asked to predict whether the bag would reach up to the same body part on an adult. A represents the child's forearm, B is the adult's forearm, and C the bag; the problem structure is: $A = C$; $A < B$; thus $C < B$.

The ability to make comparisons of two bags by measuring them against a fixed unit was clearly demonstrated by school children and by unschooled adults. Unschooled children made some errors, and only about half of them produced uniformly correct responses. In summary, transitive inferences based on measurement with a fixed unit were observed among subjects with or without schooling, although schooling seemed to speed up the process of development of the ability to make transitive inferences under these circumstances.

Greater variation in performance was observed for the second group of tasks using different-sized units. Unschooled adults outperformed school children with as much as six years of schooling—a finding that indicates schooling is not the critical variable in the ability to make transitive inferences using units of different sizes. However, unschooled adults did not perform at ceiling level. Among the unschooled adults, 64% made three or four correct predictions in response to four questions, 18% made only one or two correct predictions, and the remaining 18% made no correct predictions. This study provides clear evidence for out-of-school development of reasoning and inference-making about different-sized units. It also shows that schooling speeds up the process of development in the simpler but not the more complex task.

Carraher (1985, 1989) also tested whether illiterate Brazilian adults can make type 2 and type 4 inferences in the context of a monetary system without previous instruction in writing numbers. Monetary systems involve all the inference-making about units previously described: Type 2 inferences were evaluated by asking adults a simple question: If one person had five bills of 10 *cruzados* (Brazilian currency at the time) and a second person had five bills of 100 *cruzados,* would they have the same amount of money?

Type 4 inferences were tested by asking subjects to say how they would pay imagined amounts of money (quantities were in the hundreds) using only bills of 100, 10, and 1 *cruzados* and giving out the smallest number of bills possible. All subjects responded correctly to the type 2 inference question. Some variability in performance was obtained in the second group of questions involving type 4 inferences. The results are summarized in Figure 22.1.

Despite the fact that adults did not perform at ceiling level on this second task, results indicate that a sound ability to understand the decomposition of a value into hundreds, tens, and ones can be developed through everyday practice. This task,

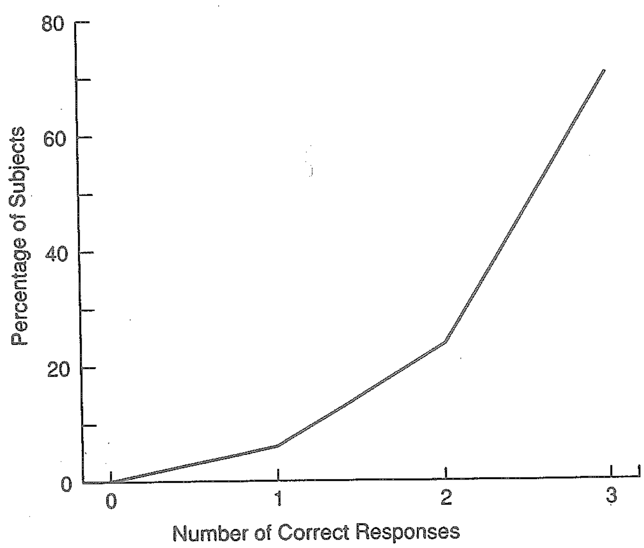

FIGURE 22–1. Percentage of unschooled Brazilian adults by number of correct responses in a three-item task analyzing numbers into hundreds, tens, and units in the context of monetary values.

which was about imagined amounts of money and involved restrictions that do not apply in the monetary system (the task used only bills of 100, 10, and 1, although the system has bills of 500, 50, and 5 that could be used), was not an everyday task but a rather complex transfer task. Even so, 70% of the subjects performed at ceiling level.

In summary, studies on counting and measuring in everyday activities indicate that subjects can reinvent the concept of grouping into units for counting in ways appropriate to their activity. Further, this ability is independent of the presence of a base structure in the particular numeration system. Everyday practices also create opportunities for individuals to make inferences typical of classroom mathematical activities.

THE ETHNOMATHEMATICS OF ARITHMETIC OPERATIONS: ORAL AND WRITTEN VARIETIES OF ARITHMETIC

In the preceding section we considered only the oral representations of numbers. In this section we will look briefly at various systems for writing numbers. Having considered quantification and the use of numeration systems for measuring and making quantitative comparisons, we will turn to the use of numbers for calculating by examining the distinction between counting and solving arithmetic problems. Finally, the differences and similarities between oral and written arithmetic will be considered.

Oral and Written Numbers

Spoken numbers may be forgotten. Written systems are a culturally devised means of handling this memory load problem. As with oral counting, written numbers are uniquely represented by individual cultures. The Romans, for example, developed a written numeration system based on five ways of representing a number: (a) variation of form (*I* for *one* and *V* for *five*), (b) repetition (*III* for *three*), (c) addition (*VII* for *seven,* which is the result of adding V and II), (d) subtraction (*XL* for *forty,* which is the result of subtracting X from L), and (e) restriction rules (*V* cannot be used with repetition; *I* and *X* are used subtractively but *V* and *L* are not). This system is ingenious but not as powerful as the Hindu-Arabic system used today.

The Hindu-Arabic system is based on two major concepts: variation of form (ten digits, each with unique meaning) and place value. In a place-value system, the relative value of a digit is indicated by its position. There is no need to explicitly state its value in terms of a base. For example, one does not have to indicate that the 2 in 23 signifies the number of tens. In contrast, current Chinese written representation of numbers uses explicit reference to the grouping or base (see Figure 22.2): The representation for the number of tens is followed by the sign for 10 and the representation for the number of hundreds is followed by the sign for 100. (This explicit reference to the base or a power of the base in written representation may facilitate the understanding of the meaning of written numbers, but I know of no research that has investigated this issue.)

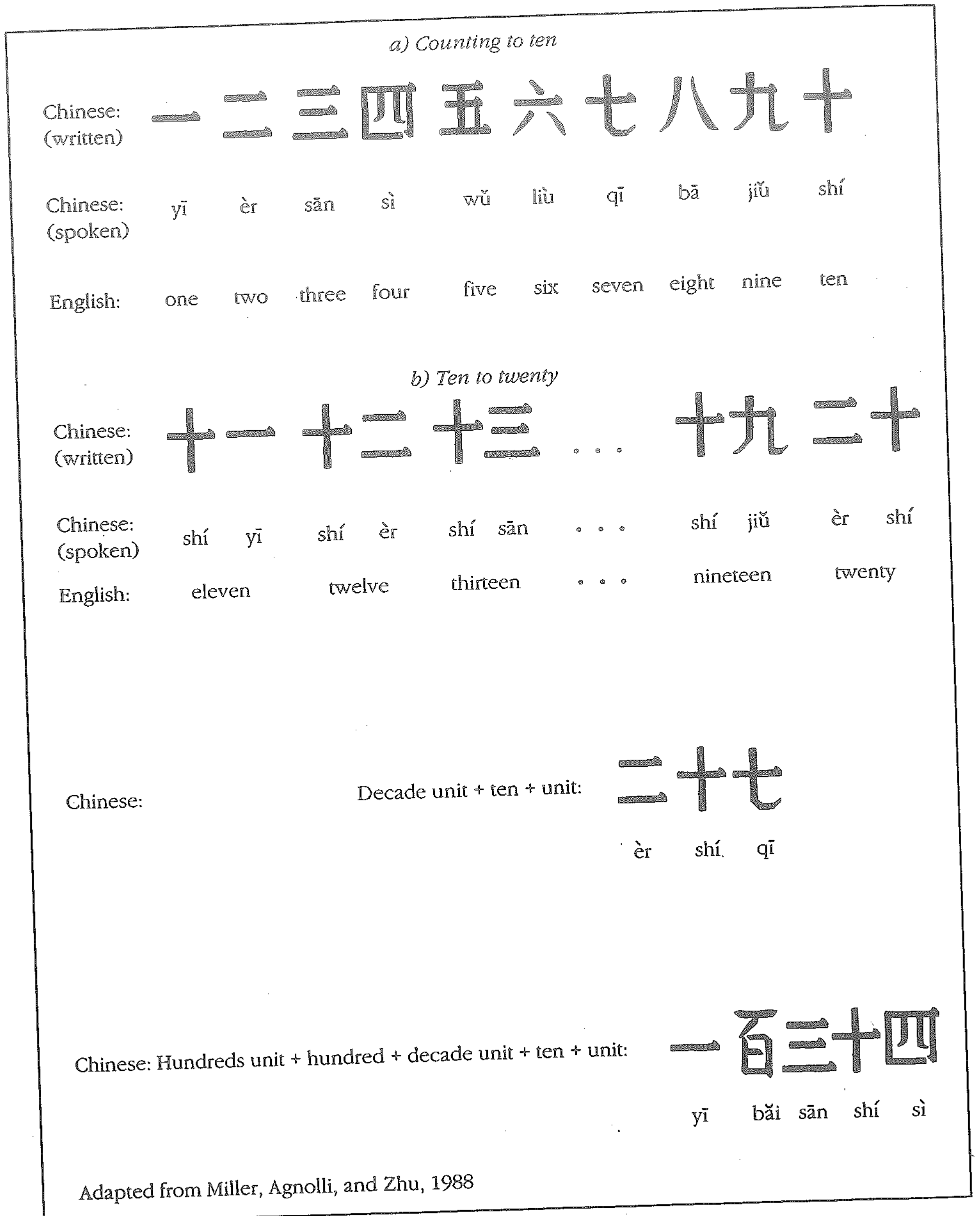

FIGURE 22–2. Oral and written numbers in English and Chinese.

Just as some oral numeration systems are more efficient for counting and measuring than others, some systems of written representation may be more efficient for calculating than others. According to Menninger (1969), Roman numbers were used to register values but not to calculate. Romans calculated by counting and using different forms of the abacus; their notation system did not facilitate calculation. In contrast, the Hindu-Arabic system, as described by Fibonacci is considered to be a calculating machine (D'Ambrosio, 1986) because it makes column arithmetic simple through positional notation and the use of zero as a place holder.

Written numeration systems vary across time and culture. Some written systems appear to make calculation easier through column arithmetic, whereas others are not useful for calculating. When the latter type of written numeration system is used, calculation tends to be carried out by counting. Across cultures, individuals differ by using concrete manipulatives to calculate or by basing calculations on written symbols. Are there common logical invariants underlying both oral and written arithmetic, or are the procedures so radically distinct that different processes underlie each?

Counting and Solving Problems

When individuals solve addition problems by putting sets together and counting the total set, are they simply counting or are they performing an arithmetic operation? If the objects were represented by fingers or pebbles, and these representations were moved about to represent the operations (for example, put together to represent addition, or separated to represent subtraction) and counted, would the subject be counting or performing an operation?

Drawing a distinction between counting and solving addition and subtraction problems is not easy. Many researchers have shown that not everyone who can count can also use counting to solve problems. The ability to use counting to solve addition and subtraction problems increases with age, varies with situation, and is influenced by schooling (see Carpenter & Moser, 1982; Fuson, 1982; Hughes, 1986; Saxe, 1985; Steffe, von Glasersfeld, Richards, & Cobb, 1983). These studies indicate that something more than counting is going on when subjects use a counting strategy in problem solving. Counting then becomes a problem-solving technique, not an activity in itself. In solving a problem by counting fingers, subjects assume (at least implicitly) that whatever results apply to fingers also apply to the original situation.

In their pioneer work on everyday mathematics among the Kpelle, Gay and Cole (1967) suggest that, despite the fact that the Kpelle solve problems by putting objects together, taking objects from sets, putting like sets of objects together, and sharing objects, "they recognize no abstract arithmetic operations as such" (p. 50). Gay and Cole were able to identify in the Kpelle language expressions for addition (for example, an expression meaning "two chickens joining three chickens is five chickens"), for subtraction ("removing two chickens from a group of three chickens leaves one chicken"), for multiplication ("three sets of two chickens are six chickens"), and for division ("putting ten bananas into five equal piles makes two bananas in each pile"). Gay and Cole described the normal procedure for solving these problems to be the use of fingers or stones to represent the objects, which were then counted. These procedures were used accurately only for problems involving small numbers. With large numbers, the procedures became cumbersome and boring, and subjects often simply guessed a large number as the answer.

Gay and Cole suggested that the Kpelle "have no occasion to work with pure numerals, nor can they speak of pure numerals. All arithmetic is tied to concrete situations" (p. 50). The Kpelle can say something that translates as "two of them added onto three of them is five of them." However, they cannot say something that would be translated as "two and three is five"; their numerals act as modifiers and cannot be used abstractly.

Similarly, Hughes (1986) found that young British children treated questions such as "If there was one brick in the box and I added two more, how many would there be?" very differently from the question "What is one and two more?" Questions about this hypothetical box with small numbers of objects were answered correctly 56% of the time whereas questions strictly about numbers were answered correctly only 15% of the time. Hughes also comments on the fact that he occasionally left out (unintentionally) the nouns being quantified by the numbers in the hypothetical situations and that as long as "they were locked into a series of questions on a particular topic, they did not need the topic spelled out to them every time.... It seemed that children's difficulties arose whenever phrases such as 'two and one' *did not refer to any specific objects*" (p. 38).

Data such as those reported by Gay and Cole (1967) and by Hughes (1986) are sometimes interpreted as indicating that some people are capable of concrete but not of abstract reasoning with numbers. It is important to check this interpretation. Using fingers as representations of something concrete—like chickens or bricks—demonstrates an abstraction. This use indicates that whatever number results from combining x fingers and y fingers will also be obtained when combining x chickens and y chickens, where x and y each represent the same number in both situations. This is undoubtedly an instance of abstraction and transfer where a technique known to work in one context is used to solve a problem in another.

A counting strategy used for addition can thus be considered a form of abstraction. Yet it is distinct from the procedure used for written calculations. In problem solving, are there any logical invariants common to counting strategies and written calculation strategies? Are the differences between oral and written calculations conceptual in nature, or are they superficial in the sense that both types of calculation rest on the same underlying principles? This question is the central focus of the next section of this paper.

Oral and Written Arithmetic

The concrete versus abstract dichotomy was prevalent in the literature about arithmetic operations in diverse cultures until Reed and Lave (1981) suggested a new way of looking at this issue. Their research consisted of a detailed analysis of arithmetic problem solving among tailors in Liberia. They began with participant observation and informal interviews in the tailors' shops. The field notes were later used in the development of experimental tasks designed to sample the four arithmetic operations at five levels of difficulty, which took into account the size of the numbers in the problems and the need to carry out regroupings. Reed and Lave describe two basic strategy types observed for solving problems.

> There are two classes of strategies for performing arithmetic operations, those that deal with quantities as such and those that deal with number names. Strategies that work with quantities are universal and manifest themselves as counting on fingers, manipulating pebbles and the like, or using an abacus. The tailors make extensive use of counters—either movable ones, such as pebbles, or marks on paper. Strategies using number names are typified by Western algorithmic manipulations learned in school, for example, "put down the 4 and carry the 1" litany in addition. In such treatment it is the manipulation of symbols, in a real sense divorced from reality, that carries the burden of computation. (p. 442)

Reed and Lave observed that different types of errors resulted from each of these two strategies. Tailors using the *manipulation-of-quantity* strategy were off by small amounts when calculating with small numbers but had great difficulty when large numbers were involved. (Modeling with pebbles when large numbers are involved becomes a less manageable procedure.) Tailors who had more schooling and had been exposed to Western algorithms made errors in carrying and borrowing, and thus were off by one 10 or one 100, but they showed less marked difference in their abilities to calculate with small versus large numbers. Reed and Lave conclude that the evidence does not favor a concrete versus abstract distinction but does demonstrate the existence of multiple arithmetic systems in a single culture.

Carraher et al. (1985), like Reed and Lave, were also able to document the existence of multiple arithmetic systems within a single culture. They began with the observation that children from lower-income families are often involved in the informal economy in large Brazilian cities; that is, these children carry out odd jobs such as washing cars, shining shoes, or selling candy and other low-priced items. In this setting, the children seemed competent to carry out mathematical computations. In contrast, children from lower-income families often fail in school mathematics when taught the numeration system and the arithmetic operations formally. In order to document this disparity, Carraher and colleagues bartered with five children selling fruits and vegetables in street markets, posing problems to them that involved addition, subtraction, and multiplication. The following week, the researchers presented each child with word problems and computation exercises involving the same numbers used during the vending interaction. The contrasting problem presentations identified two systems of arithmetic that seemed to function independently. Not only were the rates of correct responses significantly different across presentations, but also the procedures used by the children were clearly distinct. Carraher and colleagues' analysis of the protocols also indicated a qualitative difference between the two strategies used: As vendors, subjects solved the problems orally; acting as students, they solved them using paper and pencil.

Carraher et al. (1987) conducted a second study in which they were able to obtain better records of the strategies used in calculation. This study involved 16 third-grade children solving three types of problems: (1) problems presented in a simulated store, in which the child played the role of the vendor, and the experimenter the role of the customer; (2) word problems; and (3) computation exercises. Again, performance differed by presentation, and two distinct systems of arithmetic practice were observed.

Carraher et al. (1987) analyzed the oral practice in detail and described one general strategy used to solve addition and subtraction problems (*decomposition*) and one strategy for solving multiplication and division problems (*repeated groupings*). These two strategies for oral practice—which had already been documented by Ginsburg et al. (1981), Hunter (1977), and Plunkett (1979)—were analyzed in terms of their underlying mathematical properties by Resnick (1984) and Carraher and Schliemann (1988). A very simple example of decomposition can be seen in the following protocol, in which the child was solving the problem 200 − 35: "If it were 30, then the result would be 70. But it is 35. So, it's 65, 165." Discussing this example, Carraher and Schliemann suggested the following:

> The child decomposed the problem 200 − 35 into steps which seem to be the following: (1) 200 is the same as 100 + 100; (2) 100 − 30 is 70; (3) 70 − 5 is 65; finally, (4) adding the 100 which had been "set aside," as some children say, 165.... The general principles underlying the written [algorithms] and the oral strategies seem to be the same. (pp. 182–183)

In other words, written algorithms taught in school and decomposition used in oral practices rely on the same property of addition, *associativity,* referring to the fact that the way addends are grouped does not affect the sum.

Similarly, Carraher and Schliemann (1988) propose that the strategy of repeated groupings typical of multiplication and division in oral practice and the school-taught algorithms for these operations rely on the same mathematical property, *distributivity.* The use of distributivity can be clearly identified in the following example of division, observed in a word problem in which a child tried to figure out how many marbles each of three children will get if 120 are distributed evenly.

> Each one gets thirty, that will leave, three times thirty is ninety, that will leave... (E: That would leave how much?) Each one gets thirty, that leaves thirty. Then five more, that's fifteen. This leaves... fifteen. Then five more, that's fifteen. Each one gets ten and thirty, forty. (p. 184)

Carraher and Schliemann (1988) also pointed out that, despite the similarities in underlying principles, there are also striking differences between oral and written practices. Oral practices preserve the relative value of the parts of numbers that are being operated on: Hundreds are treated as hundreds; tens are treated as tens. In written algorithms, the relative values are set aside, and digits are manipulated as though they *were* units. For example, the expression *carry the one* is the same, regardless of whether what is being carried is one ten or one hundred. Oral practice is thus described as *meaning-based* in contrast to written algorithms, which are described as *rule-based.*

These characteristics of oral practice do not seem to be confined to Brazilian street vendors. They were clearly documented by Resnick (1984) in a study of an American child who discovered oral procedures for addition and subtraction before being formally instructed in written procedures. The protocol below describes this child's method when adding 152 and 149 orally.

> I would have the two 100s, which equals 200. Then I would have 50 and 40, which equals 90. So I have 290. Then plus the 9 from the 49, and the 2 from the 52 equals 11. And then I add the 90 plus the 11... equals 102. (I: 102?) 101. So I put the 200 and the 101, which equals 301. (p. 6)

Finally, Carraher and Schliemann (1988) point out that written strategies have certain characteristics that make them true cultural amplifiers (see Bruner [1966] and Cole & Griffin [1980] for a discussion of cultural amplifiers). Cultural amplifiers are resources in the culture that allow for an increase in a person's ability. In the case of computation, people who know how to add large numbers may still have difficulty when calculating orally due to the memory overload that occurs when one tries to retain the numbers and the partial results of computation in memory as calculation progresses. Through the use of writing—a cultural amplifier—memory overloads are avoided; through column arithmetic the advantages of a decimal numeration system are used to their fullest power.

Saxe (1988) further pursued the analysis of multiple systems of arithmetic among Brazilian children engaged in the informal economy and was able to demonstrate the oral nature of this practice in another way. His subjects, 23 ten- to twelve-year-old candy sellers with minimal schooling, showed less than 40% correct responses in a test of the standard orthography of numbers. This low performance in written numeracy contrasts with their high rates of correct responses in bill identification and currency comparison (about 90% correct responses), in bill arithmetic (about 70% correct responses), and in comparisons of price ratios (about 70% correct responses).

Baranes, Perry, and Stigler (1989) later attempted, without success, a replication of the findings in the study by Carraher et al. (1987) with American children. They used the same tasks (simulated store, word problems, and computation exercises), the same sets of numbers, and subjects from the same grade level (third grade). However, this type of approach to the replication of cultural influences on mathematics reasoning does not take into account the need to replicate the cultural conditions that allowed for the observations. In order to replicate findings about oral arithmetic in the simulated shop in Brazil, it would have been necessary to make sure that (a) American subjects were in fact exposed to oral arithmetic practices, (b) the natural situation simulated in the experiment was one in which oral arithmetic was likely to be observed, and (c) American children's practice of written arithmetic did not differ greatly from that of Brazilian children. Their failure to replicate the results observed by Carraher et al. (1987) is not surprising but is rather instructive. First, calculating total price and making change in American stores rarely requires the use of oral arithmetic. Most stores have cash registers that do the calculations, including calculating the change in many cases. It is a setting for machine arithmetic—a third type of arithmetic practice that deserves further analysis. Second, numbers in the Brazilian study were taken from actual bills used in the culture because problems involved reference to these bills. It is unlikely that the numbers used in the Brazilian study coincided with American denominations. Further, American children may have received more instruction on written arithmetic than Brazilian children from lower-income families attending public schools, who rarely attend preschool and only start learning written arithmetic in second grade.

In contrast to Baranes et al.'s (1989) failure to replicate differences between oral and written arithmetic in the United States, Lave (1988) reports findings about oral and written calculations among American shoppers carrying out price comparisons in the supermarket that are quite similar to those of Carraher et al. (1987). The similarities involve the types of procedures used and the differential rates of success, although the actual problems are much more difficult than those presented in the Brazilian study. The similarity of the results found by Lave (1988) and Carraher et al. (1987) is best understood by comparing the type of calculations being carried out. Situations that have similar meaning and involve similar practices are comparable across cultures, whereas those that may be superficially similar but have different meanings and involve different practices must be treated as distinct. Thus, in order to find invariance both in mathematical concepts and in cultural situations, it seems necessary to look deeper than the surface features of events.

In summary, the transition from using numbers for counting to using numbers for solving arithmetic operations is not as simple as one might expect. Not everyone who can count can solve arithmetic problems, although the solution to many simple arithmetic problems can be obtained by modeling the problem situation with concrete materials and then counting. Despite the fact that fingers or other manipulable representations are often used by children and unschooled adults in problem solving, it may be inappropriate to think of these solutions as *concrete* reasoning. They represent instances of modeling that always involves some degree of abstraction. When modeling is carried out as one-to-one correspondence and numbers are in the high tens and hundreds, counting procedures become cumbersome and boring and their efficiency in reduced. The oral practices used by Brazilian children do not use simple one-to-one correspondence and, consequently, can achieve good results with larger numbers. These practices clearly reflect an understanding of some basic properties of arithmetic operations despite the differences that result from the medium of representation used.

ETHNOMATHEMATICS OF MODELING: VARIATIONS IN FLEXIBILITY FOR PROBLEM SOLVING

I have argued that modeling of everyday situations with manipulatives involves a process of abstraction. When a situation is directly modeled, both the general number relations and some specific aspects of the situation are represented by the model. For example, to find out how many chicks will be left if nine are born and six of them run away, it is possible to represent the nine chicks by nine fingers and to represent the six chicks running away by hiding six fingers. Such direct modeling of a situation is not a particularly flexible procedure for solving problems. Suppose, for example, that one of Fahrmeier's inventory men was asked whether he could fill an order for 893 quarts of skimmed milk and, if not, to report how many quarts would be needed to fill the order. The inventory man would go into the icebox, verify that there were only 379 quarts of milk, and figure out how many quarts were needed to reach the requested 893. A direct modeling of this situation would require him to represent the 379 quarts in stock, to add some number of quarts to this until he reached the requested 893 quarts, and to count how many quarts he had added. This last count would inform him of the number of quarts needed. In formal terms, direct modeling of this problem would yield the representation $379 + x = 893$. Of course, we know that there is a quicker solution to the problem through subtraction ($893 - 379 = x$). This solution requires cognitive steps that make it different from the direct modeling procedure (see Vergnaud [1982] for an analysis of the cognitive tasks involved in inversion). The difficulty of the subtraction solution lies in the fact that it requires the subject to invert the representation of the actual situation. What he knows is that he has 379 quarts in store and he needs some more (he needs to add some) in order to come up with 893. To think of the 893 first and then to take away the 379 in stock requires an *inversion;* it requires applying an operation—subtraction—that is the inverse of what he wants to achieve—adding the number of quarts necessary to supply the 893 requested.

In many situations in everyday life, problems have a somewhat constant direction. For example, children usually know how many marbles they had at the beginning of a game and how many they won in the game. At the end of the game, they figure out how many marbles they now have. Inverting this situation, according to De Corte and Verschaffel (1987), creates difficulties from both the cognitive-relations and the social-meaning perspectives. Representations formed in everyday life refer both to the number relations and to the actual situation. A consequence of this double representation is that arithmetic problem solving learned in everyday life may not be very flexible.

Different cultural experiences may engender different degrees of flexibility for the procedures learned by the participants. In some situations, relationships may be observed consistently in one direction; in others, inversion may be practiced. Among the various cultural situations prevalent in Western cultures, schooling probably exposes individuals to inversion in the most systematic fashion. In school, pupils do not carry out actual transactions; rather, they refer to them. Numerical sentences are often written down when arithmetic problems are solved. Teachers often require children to write down the mathematical sentence that indicates the arithmetic operation leading to the correct answer. In the case of the order for 893 quarts of milk, for example, 893 − 379 would be the expected form of the solution sentence. A consequence of this emphasis in school on solving problems through the inverse operation is that schooling may well strongly influence the development of the understanding of inversion.

This section will review research bearing on the question of flexibility in problem solving and the ability to solve problems using inversion. The first group of studies analyzes word-problem solving in school and in experimental situations. The second group analyzes problem solving by adults in imagined situations that relate to their everyday work experience.

Solving Direct and Inverse Word Problems: The Effect of Schooling

Carraher (1988) carried out two studies in which adults' ability to solve problems through the inverse operation was analyzed as a function of schooling and the size of the numbers in the problem. Bearing in mind the possible difference in difficulty between problems with small and large numbers, Carraher made two predictions about adult performance. The first was that if school instruction plays an important role in the use of inverse operations, then unschooled adults should do significantly better solving direct problems rather than inverse problems, this difference being more noticeable with larger numbers. This is not a trivial statement that some things are more difficult than others or that unschooled adults have difficulty with basic arithmetic. Unschooled Brazilian adults are expected to perform at about ceiling level using direct modeling of problems for both small and large numbers because they easily employ oral mathematics in the solution of large-number problems. Since computations per se are not a great source of difficulty among unschooled adults, their performance on direct problems is expected to be high. If schooling plays a crucial role in the use of inverse operations for problem solving, unschooled adults were expected to perform at lower levels on inverse problems even with small numbers. This poorer

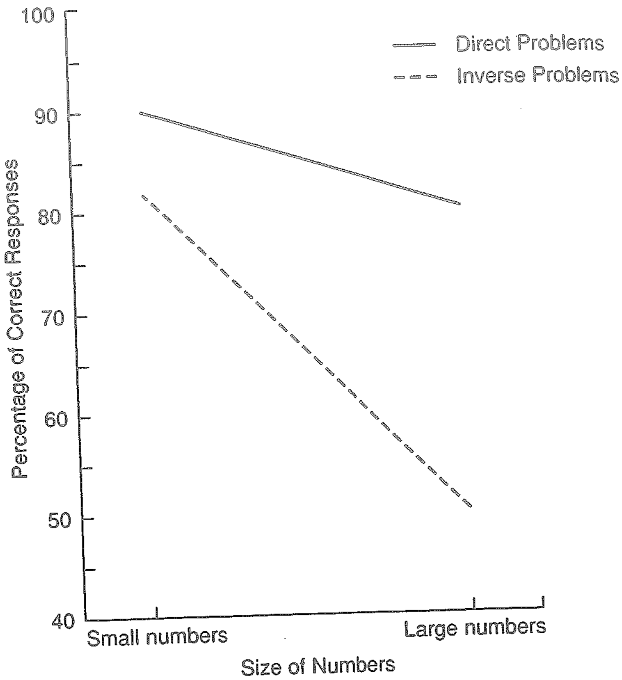

FIGURE 22–3. Percentage of correct responses by Brazilian adults enrolled in literacy programs in direct and inverse addition word problems.

performance in inverse problems would strongly contrast with the abilities displayed by American children in third grade, 80% of whom correctly solved a series of inverse addition and subtraction problems with small numbers (Riley, Greeno, & Heller, 1983). The study by Riley et al. (1983), although extremely interesting, does not offer the possibility of separating out the effects of cognitive development and schooling, which are correlated in their sample. They also used only small numbers, a methodological choice that makes it difficult to find a clear distinction between direct and inverse problems.

The second prediction in Carraher's study stated that, if schooling plays an important role in the understanding of problems to be solved through the inverse operation, then a significant effect of schooling on performance should be obtained for inverse but not for direct problems.

In order to test the first prediction, 60 adults enrolled in a literacy program (with very limited amounts of mathematics instruction) were interviewed and asked to solve six word problems, three that could be easily solved through direct modeling and three that would be more easily solved through inversion. Of the direct problems, two required addition and one required subtraction; the opposite was true of the inverse problems, which were inversions of the stories in the direct problems. Subjects solved problems with small numbers (under 20) or large numbers (above 30). A calculator was available, and all subjects were shown how to use it for each arithmetic operation before they started work on the word problems. The results of this study are summarized in Figures 22.3 and 22.4, which present the data for addition and subtraction problems, respectively. Direct problems were consistently easier than in-

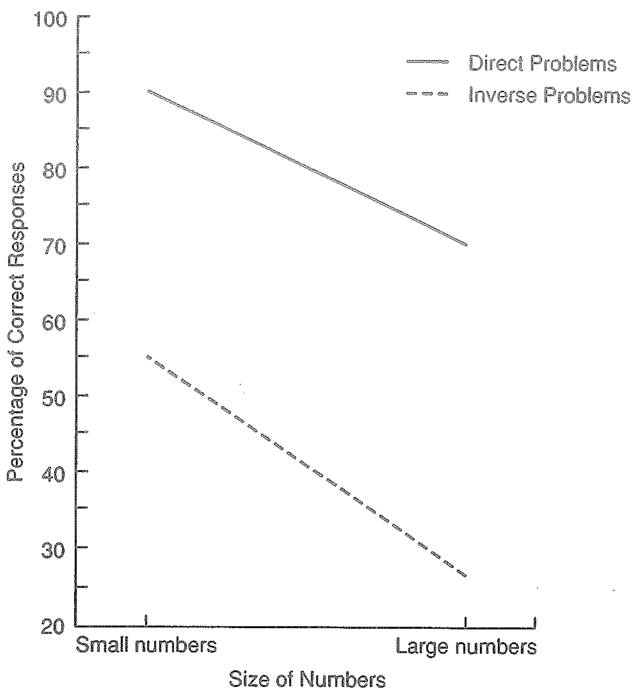

FIGURE 22–4. Percentage of correct responses by Brazilian adults enrolled in literacy programs in direct and inverse subtraction word problems.

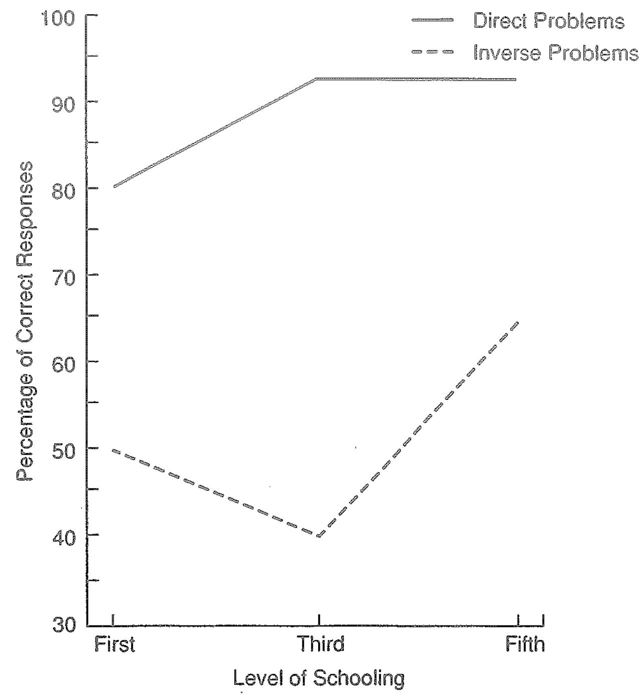

FIGURE 22–5. Percentage of correct responses by Brazilian adults enrolled in three different grade levels in night classes as a function of problem type (direct vs. inverse addition problems).

verse problems, and direct problems with large numbers were at about the same level of difficulty as inverse problems with small numbers for these subjects. The first prediction was therefore confirmed.

The second prediction, that schooling results in a significant improvement in the solution of inverse problems, was tested through a study with 90 adults enrolled in night classes. These classes were equivalent to one of three levels of schooling: first grade (basic literacy instruction), third grade, or fifth grade. Instruction on written algorithms for the four operations is completed in third grade, when students also become involved in more word-problem solving. All word problems in this study involved large numbers. A calculator was available, and its use demonstrated.

Results for the second prediction are presented in Figures 22.5 and 22.6. Direct problems remained easier than inverse problems, even with school instruction, but the effect of schooling on solving problems using the inverse operation was significant. With schooling, solving word problems becomes more flexible, and inversion is accomplished by a larger percentage of schooled than unschooled subjects.

The differences in performance just described should not lead to the conclusion that unschooled adults are concrete thinkers or that modeling is a poor resource for problem solving. The point is simply that schooling may be a source of learning about inversion, either through systematic practice using inversion in problem solving or through the introduction of symbolic systems that unify very different situations under the same symbols, like + and −. However, further research is necessary before a causal relationship between school instruction and the understanding and use of inversion can be established.

The relationship between ways of modeling situations and the ability to solve problems appears to be rather complex, and it is probably related to how the understanding of situations and solutions was initially achieved. Modeling may be helpful when there are several ways to solve a problem, some of which are formal and others unorthodox. Modeling situations is probably most helpful when the syntax for the mathematics in question is poorly understood. For example, if children are given an algebra problem to solve in the form of an algebraic sentence, like $2a + 16 = 304$, they can only solve it by manipulating the symbols according to the rules they have learned in school; there is no situation to be modeled. Had this problem been presented through words about number relations or a situational context, it might read as follows: "A number multiplied by two plus 16 equals 304; what is this number?" Under this verbal presentation, the problem can be solved by applying rules about the order in which operations must be carried out—multiplication and division are done first, addition and subtraction are only done afterward. A parallel problem can be given that does not depend exclusively on knowledge of rules. The imaginary situation could be something like the following: "The first and second graders in a school had 304 books divided between their classes, but the first graders received 16 more books than the second graders. How many books did each class receive?" This is another inverse problem; it contains a missing addend ($2a$), which when added to 16 equals 304. If students have already mastered the relationship between modeling and solving inverse problems, it would be easier for

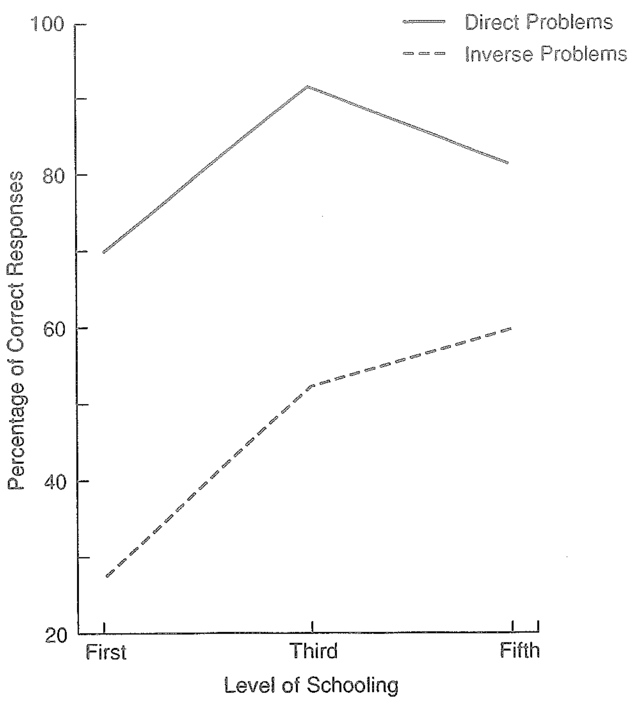

FIGURE 22–6. Percentage of correct responses by Brazilian adults enrolled in three different grade levels in night classes as a function of problem type (direct vs. inverse subtraction problems).

them to solve the problem with situational reference than with only the number sentence. Once the basic idea of modeling such situations is grasped, modeling ought to help.

This is exactly what was found by Yaroshchuk (1969), who compared the performance of a group of students solving number problems to the performance displayed by a similar group of students solving problems with the same formal mathematical description, but presented with a situational context. (The study by Yaroshchuk has a confound that should be mentioned. The numbers-only problem requires inverting two operations, addition and multiplication, whereas the problem given with situational context mentions the inverse operation of division. Since her discussion of errors indicates that the crucial step in solving the problems is the initial inversion of addition, the confound can be overlooked for the present discussion. However, a properly revised version of this study would be worth conducting.) Yaroshchuk observed 73% correctly and independently solved problems that were embedded in a situational context—a figure significantly greater than the 56% observed in number problems alone. Yaroshchuk suggested that it was *imagining* the objects that made it possible to solve the problems.

> It appears that imagining real objects makes it possible for pupils to carry out certain operations which they are unable to carry out with abstract numbers (if they are not made concrete in any way). For example, in the problem cited above, one can imagine that we set aside 16 notebooks, and then imagine that we divide the remaining notebooks into two equal piles, left and right; then to one of these piles we add the 16 notebooks set aside previously. It is obvious that being able to imagine concrete operations of transference considerably simplifies both understanding the problem's conditions and determining the mathematical operations necessary for solving it. (1969, p. 71)

Despite the fact that the study by Yaroshchuk has some flaws, it points to directions for future research that are fascinating. Problem solving that relies heavily on the learning of rules is often plagued by "buggy algorithms." This is well-known by teachers and has been clearly documented in research (Brown & Burton, 1978; Kieran, 1984; Resnick, 1982; Young & O'Shea, 1981). If students can come to understand the rules through imagining situational contexts that represent the algebraic presentation, they may be able to strengthen their understanding of the rules.

Modeling and Inversion of Problems in Everyday Life

In the preceding section, we looked at modeling and inversion in school word problems. In this section, we will look at inversion in problem situations that are close to everyday experiences. As pointed out by Lave (1988), word problems are part of the school culture and are unfamiliar to unschooled adults. The effect of schooling on the ability to solve inversion problems may not exist when situations familiar to the subjects are the context for the problem.

Two studies, one with foremen (Carraher, 1986) and one with fishermen (Schliemann & Carraher, 1990) analyzed the understanding of proportions in direct and inverse problems about everyday situations. If mathematics used in everyday situations is restricted to direct modeling, there should be an effect of inversion on problem-solving ability even for everyday problems. Furthermore, if school instruction is crucial for the understanding of inversion, level of schooling should correlate with performance in inverse problems about everyday situations.

Carraher (1986) observed that foremen use blueprints with ease in their everyday practice. Blueprints are drawn to scale so that any measurement on the blueprint is proportional to the actual size of the item represented. Foremen always know the scale they are working with because it is written on the blueprint. Although foremen may have to calculate the size of a wall when not specified on the blueprint, they never have to compute the scale used in a drawing. The direction of their practice is from a given unitary relation—the scale used in drawing—to larger values within the same scale. Carraher's research questioned whether, without indication of the scale, foreman would be able to calculate the scale used in drawing.

Carraher interviewed 17 foremen with levels of schooling ranging from 0 to 12 years (only three had studied long enough to have learned about proportions in school) on direct and inverse problems involving undisclosed familiar and unfamiliar scales. Subjects were shown the blueprints one at a time and asked to calculate the length of a specific wall x. The information they had at their disposal was characteristic of a proportions problem: Length of wall 1 (given) is to its length on the blueprint (given), as length of wall 2 (to be found) is to its length on the blueprint (given).

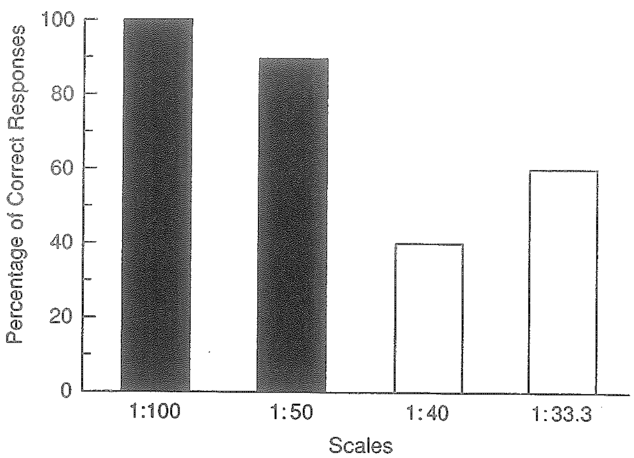

FIGURE 22–7. Percentage of correct responses by Brazilian foremen in inverse problems about scale drawings using familiar scales (1:100 and 1:50) and unfamiliar scales (1:40 and 1:33.3).

Results showed a difference in accuracy between responses for direct and inverse problems and, within inverse problems, between familiar and unfamiliar scales (see Figure 22.7). Performance on direct problems was at ceiling level. Performance on inverse problems with familiar scales was close to ceiling level, whereas only 42% and 59% of the responses to problems with unfamiliar scales were correct.

An analysis of the strategies used in the problems with unfamiliar scales showed that all the subjects who responded accurately had no trouble with inverting their procedures. These subjects would find a simplified ratio ($1/x$), which they then used to solve the problem. An example of this strategy is illustrated in the following response by J. M., an illiterate foreman with 12 years of experience on the job (working with the unfamiliar, undisclosed scale 1:40): "This one we'll have to divide. We will take 5 centimeters here and here 2 meters (data about the first wall).... This one is hard. One meter is worth 2 and a half centimeters" (p. 535).

In contrast, all but one of the subjects who failed in problems with unfamiliar scales attempted to solve the problem through the strategy of hypothesis-testing. They worked by assuming the scale was a familiar one and tested it against the known data. When they found the scale did not fit the data, they discarded that hypothesis. After unsuccessfully testing for all the scales they knew, they gave up. An example of this follows:

L.S.: (working with the undisclosed scale 1:33.3 and the data 9 cm:3 m = 15 cm:x) Nine centimeters, 3 meters. This scale is... 1 by 50, no, that would be $4\frac{1}{2}$ meters. (Pause) If you drew it like this, that is because it is correct. (Pause) Can't do it.

E: Why not? You solved all the others.

L.S.: Because it doesn't work for 1 by 50, it doesn't work for 1 by 1 [meaning 1 by 100], and it doesn't work for 1 by 20. There are three types of scale, 1 by 50, 1 by 20, and 1 by 1. The simplest scale is 1 by 1; you don't have to work on it, you look at the centimeters and you know the meters. Now, 1 by 50 and 1 by 20 you have to calculate. Now, this one here, it shows 9 centimeters by 3 meters. I've never worked with this one. I've only worked with the other three. (p. 585)

Carraher, Schliemann, and Carraher (1988) pointed out that the hypothesis-testing procedure avoids inversion. Subjects testing hypotheses still work on the problem in the habitual direction, from the hypothesized definition of the scale to the higher value. They succeed in problems with familiar scales because there are few commonly used scales: They can eliminate those scales that do not fit the data until they find the hypothesis that works. When the scale is unfamiliar, the subject gives up after testing all of the familiar possibilities.

Unlike the previous study on addition and subtraction, no correlation between schooling and facility in solving inverse problems about proportions in everyday situations was observed. Schooling's lack of effect may be due (1) to the fact that only three subjects had stayed in school long enough to be taught proportions or (2) to their inability to apply what they had learned in an academic setting to their everyday experience with scale drawing.

Schliemann and Carraher (1990) obtained similar results in another study on direct and inverse proportion problems related to everyday situations. They asked fishermen (levels of schooling ranging from no schooling to incomplete secondary school) to solve direct and inverse problems for two types of conditions: (1) price-per-kilo relationships, which they often calculate in their everyday activities, and (2) unprocessed-to-processed seafood weight ratios, a relationship that they know to be approximately fixed and proportional but that they do not use in calculation. Direct problems proved significantly easier than inverse problems. Performance on both types of inverse problems was good, but not at ceiling level: 78% of the price-kilo problems and 77% of the food-ratio problems were solved correctly. Hypothesis-testing was observed as a strategy used to avoid the difficulties with inversion. This was generally used by subjects who had failed on the inverse problems. Some subjects succeeded by adjusting wrong hypotheses in the correct direction. No correlation was observed between schooling and performance on inverse problems. These results parallel those obtained with foremen, confirming the greater difficulty of inverse problems and the absence of a schooling effect on the ability to solve inverse problems about proportions in everyday life.

In summary, modeling can be helpful in solving mathematical problems. However, there are problems more easily solved by the operation that is the inverse of the operation suggested by the semantics of the problem or by practice in everyday situations. Schooling seems to be an important factor in facilitating inversion in word problems, but no similar effect was observed for problems related to the subjects' everyday activity.

CONCLUDING REMARKS

Ethnomathematics and everyday cognition are relatively novel topics of research, but already a substantial literature has built up based on studies that vary in style, purpose, and basic assumptions about what counts as mathematics in everyday life. Other reviews (Stigler & Baranes, 1988) have been written on culture and mathematics, and whole issues of journals (*Educational Studies in Mathematics*, Bishop, May, 1988; *Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, January, 1984, and January, 1990) have been devoted to the subject. Strong criticisms of work in the area have also been written (Chevallard, 1990).

To attempt a broad coverage of such research and theoretical arguments would have forced this review to be superficial. The selection of three topics of interest in mathematics education—numeration and measurement systems, problem solving and computation, inversion and modeling—allowed for analysis of the mathematical invariants embedded in everyday situations involving these three concepts. This choice was guided by a theoretical position that identifies as mathematical those activities in which people represent objects and events in ways that allow them to extend their knowledge through mathematical inferences. It was shown here that, despite the existence of great variation in the forms of mathematical representation across cultures, the processes involved in making inferences and the types of mathematical properties used in making these inferences do not differ.

In this concluding section, we will address two general questions: (1) What sort of theories about the development of mathematical concepts can incorporate the data available on culture and mathematics learning? (2) What are the implications for mathematics education based on findings and theories about everyday mathematics?

Theoretical Generalizations

Three concepts taken from current theories about cognitive development are useful for understanding the results we have been discussing.

Cultural Amplifiers. First, we must consider the concept of cultural amplifier, initially proposed by La Barre (1954) and introduced in developmental psychology by Bruner (1966). According to this concept, the greatest changes determining man's adaptation in the last half-million years since the human brain acquired its present size and morphology are the culturally devised means to amplify human action (such as cars and hammers), sensory systems (such as radars), and reasoning skills (such as language and mathematical representation). As noted earlier, numeration systems amplify the human capacity to count and register numbers beyond the limits imposed by human memory. Place-value systems increase the range of counting and calculation beyond what is possible through manipulation of objects or oral calculation. Mathematical representational systems do not simply express meanings that people already have in mind. These systems enhance human knowledge through the use of culturally defined conventions and underlying logical structure.

Culturally defined conventions may vary across cultures and across situations in the same culture. For example, measuring systems have units of different values embedded in systems with different organizations both within and across cultures. These differences influence how people think about the measured dimension and the number relations that are familiar to them. Distances are *thought of* in terms of miles in countries that measure with miles and in terms of kilometers in countries that use kilometers.

The similarities underlying the representational systems in mathematics are perhaps more important than the differences. All measurement systems, for example, use the concept of unit and allow for inferences of the same form based on the concept of unit, regardless of the size of the unit, and its relationship with other units of measurement. Finally, what is most important is that all of these conventional symbolic systems work as cultural amplifiers for their users, amplifying their abilities more or less as a function of the systems themselves.

Socialization of Thought. A second important idea taken from the study of cognitive development that helps to analyze the relationship between culture and mathematical thinking is provided by Piaget (1962) and Vygotsky (1978), who recognized the socializing role of conventional representational systems in their analysis of natural language. Mathematical representation, like natural language, is a conventional and collective system of signifiers, the meaning of which must be provided by the subject. With respect to language, Piaget says,

> Representation is thus the union of a "signifier" that allows recall, with a "signified" supplied by thought. In this respect, the collective institution of language is the main factor in both the formation and socialization of representations, but the child's ability to use verbal signs is dependent on the progress of his own thought. (1962, p. 273)

The socialization of representations in mathematical reasoning involves both the use of the same symbols (the same numeration system, the same algebraic notation, and so forth) and the use of the same boundaries for the classification of concepts (the same set of situations classified as "addition problems," for example). Luria (1966) emphasized the role of language in the establishment of common boundaries for concepts by indicating that, when the mother points at an object and says a word to refer to it, the object becomes a figure against the background. In this process, the essential functional properties of the object become more salient as it is placed in the same category as other objects designated by the same word. Similarly, mathematical representation brings, as we saw earlier, heterogeneous situations under the same designation. Children must provide the meaning for signifiers such as + (plus) or − (minus) by finding the mathematical properties common to the situations designated by the signifier. However, as Resnick (1984) pointed out, mathematical objects cannot be pointed to as the referents of natural language concepts (chair, cup, and so on) often can. You can point to three chairs and say "three" or point to the symbol 3 and say "three," but that is not pointing to a referent for the understanding of the concept of number. The task that mathematics educators have to accomplish is the socialization of mathematical reasoning. This is similar to the mother's task in the socialization of the child's representations through natural language, only more difficult.

The meaning of a mathematical concept is always abstract, and its acquisition is represented by the understanding of relations and invariants, not by the recognition of physical objects. For example, proportion problems may involve situations that vary substantially from the viewpoint of their content. These problems may involve the relationship between the number of coconuts purchased and their cost, or the relationship between raw shrimp and processed shrimp, or the relationship in a recipe between amount of flour and amount of milk. On the surface, these situations are very different, yet they share the same underlying logic. All of these problems involve at least two variables and, as one variable fluctuates, the other fluctuates proportionally in a one-to-many correspondence. In order to attribute meaning to the expression *proportion problem,* pupils must understand what is common to these situations, what is invariant in the relationship between the variables regardless of their values. This understanding of the invariants in proportional situations represents the core meaning of proportion problems. If this understanding is achieved, individuals can make changes in the values of variables without altering their relationship. For example, a shopper can buy additional coconuts without changing the unit-price relation, or a cook can change amounts of ingredients in a recipe in order to serve more guests.

Cognition and Metacognition. The third useful concept in the literature about the development of mathematical concepts relates to the distinction between cognition and metacognition. This distinction has been used in many contexts—for example, distinctions have been made between memory and metamemory (Flavell & Wellman, 1977) and between linguistic and metalinguistic knowledge (Sinclair, Jarvella, & Levelt, 1978). In general, the simple concept refers to the ability that a subject has to use a specific type of knowledge as a tool, whereas the metaconcept refers to the subject's ability to use that same piece of knowledge as an object in thinking. For example, native speakers of a language make the phonemic distinctions of their language when speaking and listening. They use phonemes as tools because phonemes are the abstract units of pronunciation. They recognize the smallest distortions in phonemes when they are, for example, pronounced by foreigners. However, the ability to isolate and count phonemes within syllables cannot be assumed for all native speakers of the language. The development of metalinguistic knowledge is closely associated not with speaking and listening but with other specific cultural practices, such as learning about rhymes and with acquiring written language. Simple concepts and metaconcepts seem to develop in association with distinct types of cultural practice.

Mathematical concepts have already been subjected to a tool-object distinction by Douady (1985). As a tool in everyday life, mathematical concepts are used in the process of structuring external situations as the subject tries to understand the mathematical relations involved in these situations. But the concepts remain in a sense transparent when they are used as tools. The subject thinks about the situation, not about the mathematical concept. From this perspective, mathematical concepts in everyday life are not restricted to concrete thinking and are not mere procedures replicated without understanding.

As the object of instruction, mathematical concepts are the focus of interest in the classroom. Pupils are expected to learn, think about, and try representing the concepts themselves. In this sense, everyday mathematics and mathematics education are clearly distinct. Mathematical concepts are tools in everyday life and objects in the classroom. Mathematical concepts are occasionally used as tools in the classroom in word problems and other application exercises, but the focus of interest even in application exercises is the mathematical concept, not the particular situation.

Implications for Education

Does the difference between everyday and school knowledge of mathematics mean that everyday mathematics is of no use for classroom instruction? Of course not. According to developmental psychologists, in order to succeed in studying a mathematical concept as an object, the subject must already have access to the concept as a tool. Karmiloff-Smith (1988), who attributes great importance to the relationship between cognition and metacognition in cognitive development, makes this point clear.

> Finally, I want to take it that the human mind not only tries to appropriate the *external* reality that is pre-set to explore and represent, but that it also tries to appropriate its own internal reality. What I mean by this is that the human mind re-represents recursively its own *internal* representations, thereby creating meta-procedures and meta-representations in general. For a long time I've argued that it is this *recursive* capacity to represent one's own representations that sets humans apart from other species. (p. 12)

If we accept for the moment the thesis that metaknowledge is the representation of one's own internal reality and that concepts must first be used in the representation of external reality in order to be transformed into metaconcepts, some directions can be extracted for mathematics education from the analysis of cultural practices embedding mathematical knowledge.

When a teacher faces a class with the aim of teaching a particular concept, the teacher must ponder whether the pupils are likely to have used the concept in everyday life. Until more research is available on this subject, teachers will have the task of looking for everyday uses of the concept and attempting to analyze the logical invariants that underlie everyday concepts and procedures. Everyday procedures will often differ from school-taught procedures, and teachers will need to look beneath the surface to recognize which invariants are being respected in everyday practice. I am hopeful that the studies reviewed here can provide a model for this enterprise.

After identifying an everyday situation in which the concept is used, the teacher will need to know how to use the situation to promote an awareness and understanding of the concept. Research on this process is scarce. The suggestions available in the literature are of two types. First, the *abstraction-generalization* model can be called into play in the learning-to-learn context (see Cole, 1977) or in the cognitive apprenticeship approach (see Brown, Collins, & Duguid, 1989). The basis of these approaches is that understanding several different situations involving the same invariant leads to the abstraction and generalization of the core concept (the invariant), and to the enrichment of the concept by extending the set of situations to which it applies. In contrast to present practice, this model of teaching proposes that pupils solve several problem situations embedding the same concept and then turn to the concept as such. For example, pupils could be given several problems about fractions without being taught how to represent fractions, and then they might connect the meanings developed in this way to the mathematical representation. This model of teaching was attempted with success by Lima (1989) in the context of fractions.

Second, the rerepresentation of the original situation through natural language can help the transition of a concept from tool to object. In this case, situations become data to be talked about rather than problems to be solved. Carraher and Ruiz (1985) attempted such a method in a study of the development of the concept of proportions. In their procedure, pupils worked in small groups to solve a problem about proportions. They were told that a second group of students would be solving an identical problem with different numbers. They were asked to write a message to this second group telling them how to solve the problem. However, since they did not know what the numbers would be in the other group's problem, they had to explain the procedure without mentioning numbers. They were told that their success in the task was not a matter of whether they had solved the problem but whether their message was useful to the other group. Under these circumstances, students realized that the message had to be a general one about the relationships between the variables, not a specific instruction for calculating, such as, "multiply 2 by 13 and divide by 3." Most students attempted to write the general message in natural language; a few attempted to devise a formula by writing a message such as "Make x your number of rings on the scale and y the number of the peg on which you put your rings." Not all the groups who could solve the problem could write general messages successfully, which showed that rerepresentation of the problem was not easy. Further studies are needed to verify what effect this type of reconstruction of the problem may have on pupils' understanding of the mathematical concept.

In summary, this chapter has attempted to show that mathematical activity can be observed as interwoven with everyday practice outside academic settings. Mathematizing objects and situations in everyday activity means representing them in ways that allow for the extraction of further information about the objects or situations on the basis of the representations without the need for further verification by returning to the represented objects. Three topics were analyzed in detail in the review of research: (a) counting and measuring, (b) problem solving and calculation, and (c) modeling and inversion. These concepts are central to the types of mathematical activity engaged in by pupils at the elementary-school level. They are also of great importance for everyday life and become interwoven in everyday activities in different forms. Some of these concepts may be acquired without schooling, but this does not mean that mathematical instruction should leave learning of these concepts for apprenticeship outside school. It is the relevance of these concepts both to mathematical activity and to everyday life that suggests the need for paying greater attention to them in mathematics teaching. By promoting their understanding in school, teachers may help children recognize how much mathematics they can learn and use outside school. It might also help children transform concepts they now use as tools into metaconcepts they can use for generalization; in the process, perhaps they will "learn to learn" mathematics and enjoy it.

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